Published by Institute of Physics Publishing for SISSA/ISAS
Received: *July 19, 2004*
Accepted: *July 20, 2004*

Topological string partition functions as polynomials

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Abstract: We investigate the structure of the higher genus topological string amplitudes on the quintic hypersurface. It is shown that the partition functions of the higher genus than one can be expressed as polynomials of five generators. We also compute the explicit polynomial forms of the partition functions for genus 2, 3, and 4. Moreover, some coefficients are written down for all genus.

Keywords: Topological Strings, Differential and Algebraic Geometry.

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1. Introduction

The topological string theory on a Calabi-Yau 3-fold is a good toy model as well as a computational tool of the superstring. Especially, it is a practicable problem to calculate the higher genus amplitudes, while those of the physical string are technically difficult to calculate. The topological A-model partition functions on a Calabi-Yau 3-fold M is defined for genus $g \geq 2$ as

$$F_g^{\text{A-model}}(t) = \sum_{d \in H_2(M,\mathbb{Z})} e^{-d \cdot t} \int_{\overline{\mathcal{M}}_{g,d}} 1\,, \tag{1.1}$$

where $\overline{\mathcal{M}}_{g,d}$ is the compactified moduli space of degree d holomorphic map from genus g Riemann surface to M, and t is a vector of complexified Kähler parameters. In the physical view point, the partition function can be defined similar way to the bosonic string as follows. Let us consider the A-twisted theory of the sigma model on the Calabi-Yau M. It becomes a $N = 2$ topological CFT which depends on the Kähler parameter t. We denote by $\mathcal{M}_g$ the moduli space of the complex structure on genus g Riemann surface. Then the

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genus $g \geq 2$ topological string partition function can be defined as

$$F_g(t,\bar{t}) = \int_{\mathcal{M}_g} \left\langle \prod_{k=1}^{3g-3} \left(\int G^- \mu_k \right) \left(\int \bar{G}^- \bar{\mu}_k \right) \right\rangle_{g,t} , \tag{1.2}$$

where μ_k's are the Beltrami differentials and $G^-, \bar{G}^-$ are "b-ghosts" of $N = 2$ topological CFT. The two definition are connected by a gauge transformation $f(t)$ and the limit, namely

$$F_g^{\text{A-model}}(t) = \lim_{\bar{t}\to\infty} f(t)^{2-2g} F_g(t,\bar{t}) \,. \tag{1.3}$$

The non-holomorphic partition function $F_g(t,\bar{t})$ has a good global properties.

The $\bar{t}$ dependence is governed by the holomorphic anomaly equation which is written down by Bershadsky, Cecotti, Ooguri and Vafa [1, 2]. This equation provides an effective method to calculate the higher genus amplitudes. But there remain some ambiguities to determine the amplitudes by using the holomorphic anomaly equation. Ref. [2] have used geometric consideration to fix the ambiguity, and obtain genus 2 partition function for the quintic hypersurface. As Ghoshal and Vafa have pointed out in [3], comparing the conifold limit and the topological string on conifold gives non-trivial information. In [4], Katz, Klemm, and Vafa have used the M-theory picture and obtained genus 3 and 4 partition function for the quintic. In order to proceed this calculation, we want to understand the structure of the higher genus amplitudes.

Higher genus amplitudes are expected to have the property similar to the modular forms. Every modular form can be written in a quasi-homogeneous polynomial of Eisenstein series E_4 and E_6. This is the very beginning of the interesting theory of the modular forms. It will be interesting as well as useful if the topological string partition function has this kind of polynomial structure. Actually, as pointed out in [5], a discrete group similar to $\mathrm{SL}(2,\mathbb{Z})$ but not the same, act to the moduli space of the quintic hypersurface.

In this paper, we explore the structure of the higher genus amplitudes of the quintic hypersurface. We will show that the topological string partition function F_g can be written as a degree $(3g-3)$ quasi-homogeneous polynomial of five generators V_1, V_2, V_3, W_1, Y_1, where we assign the degree $1, 2, 3, 1, 1$ for V_1, V_2, V_3, W_1, Y_1 , respectively. The generators V_1, V_2, V_3, W_1, Y_1 are the functions of the moduli parameter whose explicit forms are summarized in eqs. (3.35).

This fact provides a simple expression of the partition function of each genus; This polynomial expression is completely closed and includes all the data of the coefficients of instanton expansion. The polynomial is also more compact than the raw Feynman diagram expression; The number of terms grows only in power of the genus.

The construction of this paper is as follows. In section 2, we review the method of calculation of topological string amplitudes by using the mirror symmetry and the holomorphic anomaly equation. In section 3, we prove that the partition function can be written as a polynomial of the five generators. Some of the coefficients are calculated in section 4. Section 5 is devoted to conclusions and discussions. In appendix A, polynomial form of genus 3 and 4 partition function are written. In appendix B, we discuss the generalization to the Calabi-Yau hypersurfaces in weighted projective spaces treated in [6].

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2. Calculation of topological string amplitudes by mirror symmetry and the holomorphic anomaly equation

In this section, we review the calculation of the topological string amplitudes by the mirror symmetry [5, 1, 2]. First, we explain the genus zero amplitudes following [5]. After that, we will explain the genus one amplitudes and the higher genus ones following [1, 2]. In this paper, we mainly work with the quintic hypersurface in $\mathbb{C}P^4$. For this reason, we will concentrate to the case of quintic in the review in this section.

2.1 Genus zero

Let us review the genus zero amplitudes of the quintic [5]. The mirror manifold of the quintic is expressed by the orbifold of the hypersurface in $\mathbb{C}P^4$ [7]

$$p := x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - 5\psi x_1 x_2 x_3 x_4 x_5 = 0\,, \tag{2.1}$$

where $x_j,\ j = 1,2,3,4,5$ are the homogeneous coordinates of $\mathbb{C}P^4$, and ψ is the moduli parameter. The orbifold group is $(\mathbb{Z}_5)^3$. If we denote the generators of this $(\mathbb{Z}_5)^3$ by g_1, g_2, g_3, the action can be written as

$$g_j : x_j \to e^{\frac{2\pi i}{5}} x_j\,, \qquad x_5 \to e^{\frac{-2\pi i}{5}} x_5\,, \qquad x_i \to x_i,\ (i \neq j, 5)\,. \tag{2.2}$$

We fix the gauge to the standard one in which the holomorphic 3-form Ω is written as

$$\Omega = 5\psi \frac{x_5 dx_1 \wedge dx_2 \wedge dx_3}{\partial p / \partial x_4}\,. \tag{2.3}$$

In this gauge, the Picard-Fuchs equation for a period $w = \int \Omega$ is given by

$$\left\{ (\psi\partial_\psi)^4 - \psi^{-5} (\psi\partial_\psi - 1)(\psi\partial_\psi - 2)(\psi\partial_\psi - 3)(\psi\partial_\psi - 4) \right\} w = 0\,. \tag{2.4}$$

There is a solution ω_0 which is regular at $\psi \to \infty$. This solution is expressed by the expansion in ψ^{-5} as

$$\omega_0(\psi) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5 (5\psi)^{5n}}\,. \tag{2.5}$$

In order to write down the other solutions, we extend the definition of ω_0 to the function of ψ and ρ to use the Frobenius argument. This function $\omega_0(\psi,\rho)$ should be the form

$$\omega_0(\psi,\rho) = \sum_{n=0}^{\infty} \frac{\Gamma(5(n+\rho)+1)}{\Gamma(n+\rho+1)^5 (5\psi)^{5(n+\rho)}}\,. \tag{2.6}$$

The natural basis of the solutions are written as

$$\Pi = \begin{pmatrix} w_0 \\ w_1 \\ \partial_1 F_0 \\ \partial_0 F_0 \end{pmatrix} = \left(\frac{2\pi i}{5}\right)^3 \left. \begin{pmatrix} \omega_0(\psi,\rho) \\ \frac{1}{2\pi i}\partial_\rho \omega_0(\psi,\rho) \\ \frac{5}{2}\left(\frac{1}{2\pi i}\partial_\rho\right)^2 \omega_0(\psi,\rho) \\ -\frac{5}{6}\left(\frac{1}{2\pi i}\partial_\rho\right)^3 \omega_0(\psi,\rho) \end{pmatrix} \right|_{\rho=0}\,. \tag{2.7}$$

This basis are standard symplectic basis. Therefore, the Kähler potential K of the moduli space can be written as

$$e^{-K} = -i\Pi^\dagger \Sigma \Pi, \qquad \Sigma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.8}$$

This matrix Σ is the ordinary symplectic bilinear form. The metric of the moduli space is obtained as $G_{\psi\bar{\psi}} = \partial_\psi \bar{\partial}_{\bar{\psi}} K$.

Let us denote the complexified Kähler parameter in the A-model picture by t. The relation between ψ and t ("mirror map") is given by

$$t = -2\pi i \frac{w_1}{w_0} = -\log(5\psi)^{-5} - \frac{5}{\omega_0} \sum_{m=1}^{\infty} \frac{(5m)!}{(m!)^5 (5\psi)^{5m}} \left(\Psi(1+5m) - \Psi(1+m)\right), \tag{2.9}$$

where $\Psi(x) := \partial_x \log \Gamma(x)$.

Another important observable is the Yukawa coupling. It is determined by

$$C_{\psi\psi\psi} = \Pi^T \Sigma \partial_\psi^3 \Pi \,. \tag{2.10}$$

This equation and the Picard-Fuchs equation read the following differential equation for the Yukawa coupling.

$$\partial_\psi C_{\psi\psi\psi} = \frac{2\psi^{-1} + 4\psi^3}{1 - \psi^5} C_{\psi\psi\psi} \,. \tag{2.11}$$

This differential equation can be solved as

$$C_{\psi\psi\psi} = \frac{(2\pi i)^3}{5^3} \frac{\psi^2}{1 - \psi^5} \,, \tag{2.12}$$

where the normalization is fixed by the asymptotic behavior. The Yukawa coupling in the t-frame becomes

$$C_{ttt}^{\text{A-model}} = \left(\frac{(2\pi i)^3}{5^7} \omega_0^2 \right)^{-1} \left(\frac{\partial \psi}{\partial t} \right)^3 C_{\psi\psi\psi} \,. \tag{2.13}$$

The first factor in the right-hand side comes from the gauge transformation, and the second factor is the contribution of the coordinate transformation. The $C_{ttt}^{\text{A-model}}$ gives the instanton expansion of the A-model picture and includes the information of the number of rational curves in the quintic.

These quantities, Kähler potential, metric, and Yukawa coupling are essential to compute the higher genus amplitudes.

2.2 Genus one and higher

The one point function $\partial_\psi F_1$ of genus one satisfies the holomorphic anomaly equation [1]

$$\bar{\partial}_{\bar{\psi}} \partial_\psi F_1 = \frac{1}{2} C_{\psi\psi\psi} \overline{C}_{\bar{\psi}}^{\psi\psi} - \left(\frac{\chi}{24} - 1 \right) G_{\psi\bar{\psi}} \,, \qquad \chi = -200 \,, \tag{2.14}$$

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where $\overline{C}_{\bar\psi}^{\psi\psi}$ is defined as

$$\overline{C}_{\bar\psi}^{\psi\psi} := \overline{C}_{\bar\psi\bar\psi\bar\psi}(G_{\psi\bar\psi})^{-2}e^{2K}\,. \tag{2.15}$$

Eq. (2.14) can be solved as

$$\partial_\psi F_1 = \frac{1}{2}\partial_\psi \log\left[(G_{\psi\bar\psi})^{-1}\exp\left(\frac{62}{3}K\right)\psi^{62/3}(1-\psi^5)^{-1/6}\right]. \tag{2.16}$$

The holomorphic ambiguity is fixed by the asymptotic behavior. In the t-frame, and topological limit ($\bar t\to\infty$), this one point function becomes

$$\partial_t F_1^{\text{A-model}} = \lim_{\bar t\to\infty}\frac{\partial\psi}{\partial t}\partial_\psi F_1\,. \tag{2.17}$$

This function gives the instanton expansion in A-model picture, and includes the information of the number of elliptic curves in the quintic.

Let us turn to the $g\geq 2$ amplitudes. First, we introduce some notations. We denote the vacuum bundle by L. Holomorphic 3-form Ω is a section of L. For a section of L, the action of the gauge transformation (Kähler transformation) are parametrized by a holomorphic function $f(\psi)$ and expressed as

$$K(\psi,\bar\psi)\to K(\psi,\bar\psi)-\log f(\psi)-\log\bar f(\bar\psi)\,,\qquad \Omega\to f(\psi)\Omega\,. \tag{2.18}$$

The genus g partition function F_g is a section of L^{2-2g} and transform as $F_g\to f(t)^{2-2g}F_g$. Besides this symmetry of Kähler transformation, there is another gauge symmetry — the reparametrization of the moduli. We will define the covariant derivative D_ψ for these two gauge transformations. If $h(\psi)$ is a section of $(T^*)^m\otimes L^n$, the covariant derivative of h is defined as

$$D_\psi h = \partial_\psi h + m\Gamma^\psi_{\psi\psi}h + n(\partial_\psi K)h\,, \tag{2.19}$$

where $\Gamma^\psi_{\psi\psi} = -(G_{\psi\psi})^{-1}\partial_\psi G_{\psi\psi}$ is the Christoffel symbol.

Next, we consider the holomorphic anomaly equation. The holomorphic anomaly equation for the genus g partition function F_g is given by [2]

$$\bar\partial_{\bar\psi}F_g = \frac{1}{2}\overline{C}_{\bar\psi}^{\psi\psi}\left(D_\psi D_\psi F_{g-1}+\sum_{r=1}^{g-1}D_\psi F_\psi D_\psi F_{g-r}\right). \tag{2.20}$$

A solution of (2.20) is given by the Feynman rule as in [2]. We denote this solution by $F_g^{(FD)}$. The Feynman rule is composed of two kind of things: propagators and vertices. We begin with the propagators. It is useful to introduce the following quantities.

$$S^{\psi\psi} = \frac{1}{C_{\psi\psi\psi}}[2\partial_\psi\log(e^K|f|^2)-\partial_\psi\log(|v|^2G_{\psi\bar\psi})]\,, \tag{2.21a}$$

$$S^{\psi} = \frac{1}{C_{\psi\psi\psi}}\left[\left(\partial_\psi\log(e^K|f|^2)\right)^2 - v^{-1}\partial_\psi\left(v\partial_\psi\log(e^K|f|^2)\right)\right], \tag{2.21b}$$

$$S = \left[S^\psi-\frac{1}{2}D_\psi S^{\psi\psi}-\frac{1}{2}(S^{\psi\psi})^2C_{\psi\psi\psi}\right]\partial_\psi\log(e^K|f|^2)+\frac{1}{2}D_\psi S^1+\frac{1}{2}S^{\psi\psi}S^\psi C_{\psi\psi\psi}\,, \tag{2.21c}$$

where f is a holomorphic section of L and v is a holomorphic vector field on the moduli space. In our gauge, we can set $f = \psi$ and $v = 1$. The quantities $S^{\psi\psi}, S^{\psi}, S$ in (2.21) are determined to satisfy the relations

$$\overline{C}_{\bar{\psi}}^{\psi\psi} = \bar{\partial}_{\bar{\psi}} S^{\psi\psi}\,, \qquad S^{\psi\psi} = (G_{\psi\bar{\psi}})^{-1}\bar{\partial}_{\bar{\psi}} S^{\psi}\,, \qquad S^{\psi} = (G_{\psi\bar{\psi}})^{-1}\bar{\partial}_{\bar{\psi}} S\,. \tag{2.22}$$

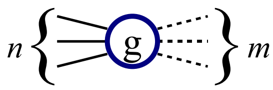
To show these relations, we use the special geometry relations [8]

$$\begin{aligned} \bar{\partial}_{\bar{\psi}} C_{\psi\psi\psi} &= 0\,, \qquad \partial_{\psi}\overline{C}_{\psi\psi\psi} = 0\,, \\ R_{\psi\bar{\psi}\psi}{}^{\psi} &:= -\bar{\partial}_{\bar{\psi}}\Gamma^{\psi}_{\psi\psi} = 2G_{\psi\bar{\psi}} - C_{\psi\psi\psi}\overline{C}_{\bar{\psi}}^{\psi\psi}\,. \end{aligned} \tag{2.23}$$

There are three types of propagators; The one connecting two solid lines, the one connecting solid and dashed lines, and the one connecting two dashed lines. The value of these propagators are written as

$$= -S^{\psi\psi}\,, \qquad = -S^{\psi}\,, \qquad = -2S\,. \tag{2.24}$$

Let us turn to the vertices. A vertex is labeled by three integers g, n, m. n solid lines and m dashed lines end to the vertex labeled by g, n, m. We denote the value of the vertex as

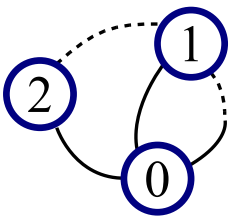


$$= \lambda^{2g-2+n+m}\widetilde{C}^{(g)}_{\psi^n,\varphi^m}\,, \tag{2.25}$$

where λ is the topological string coupling constant. The detail of the vertex $\widetilde{C}^{(g)}_{\psi^n,\varphi^m}$ are described as

$$\begin{aligned} \widetilde{C}^{(g)}_{\psi^n,\varphi^{m+1}} &= (2g-2+n+m)\widetilde{C}^{(g)}_{\psi^n,\varphi^m}\,, \qquad \widetilde{C}^{(g)}_{\psi^n,\varphi^0} = C^{(g)}_{\psi^n}\,,\ (2g-2+n \geq 1)\,, \\ C^{(g)}_{\psi^n} &= D^n_{\psi}F_g,\ (g \geq 2)\,, \qquad C^{(1)}_{\psi^n} = D^{n-1}_{\psi}\partial_{\psi}F_1\,, \qquad C^{(0)}_{\psi^n} = D^{n-3}_{\psi}C_{\psi\psi\psi} \\ \widetilde{C}^{(1)}_{\varphi} &= \frac{\chi}{24} - 1, \quad \widetilde{C}^{(1)}_{\psi^0} = 0\,, \qquad \widetilde{C}^{(0)}_{\psi\psi\varphi^m} = \widetilde{C}^{(0)}_{\psi\varphi^m} = \widetilde{C}^{(0)}_{\varphi^m} = 0\,. \end{aligned} \tag{2.26}$$

The value of a diagram is obtained by multiplying the values of all the elements. For example, the following diagram is the one which contribute to F_5. We can evaluate this diagram as



$$= \lambda^8\widetilde{C}^{(2)}_{\psi,\varphi}\widetilde{C}^{(0)}_{\psi\psi\psi}\widetilde{C}^{(1)}_{\psi,\varphi\varphi}(-S^{\psi\psi})^2(-S^{\psi})(-2S)\,. \tag{2.27}$$

The Feynman diagram part of the partition function is the sum of all connected diagrams $\mathcal{D}$ divided by the appropriate symmetric factor (constant) $s_{\mathcal{D}}$

$$F_g^{(FD)}\lambda^{2g-2} = -\sum_{\substack{\mathcal{D}:\text{connected diagrams} \\ \text{of order } \lambda^{2g-2}}} s_{\mathcal{D}}^{-1}\,\mathcal{D}. \tag{2.28}$$

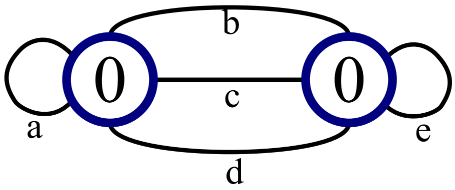


Figure 1: A diagram which contribute to F_4 the symmetric factor is $s_{\mathcal{D}} = 2 \times 2 \times 3! \times 2 = 48$.

Here, the symmetric factor $s_{\mathcal{D}}$ is the order of the symmetry group of the diagram $\mathcal{D}$. For example, figure 1 is a diagram which contribute to F_4. The symmetric factor is counted as follows; factor 2 from the exchange of the two ends of the line a, factor 2 from the exchange of the two ends of the line e, factor 3! from the interchange of the three lines b, c, d, and factor 2 from the left-right flip. The symmetric factor becomes $s_{\mathcal{D}} = 2 \times 2 \times 3! \times 2 = 48$.

Finally let us mention the holomorphic part. The general solution of the holomorphic anomaly equation (2.20) is

$$F_g(\psi, \bar{\psi}) = F_g^{(FD)}(\psi, \bar{\psi}) + f_g(\psi)\,, \tag{2.29}$$

where $f_g(\psi)$ is a holomorphic function. From the asymptotic behavior, $f_g(\psi)$ can be written as the following form.

$$f_g(\psi) = \sum_{j=1}^{[(2g-2)/5]} b_{g,j} \frac{1}{\psi^{5j}} + \sum_{j=0}^{2g-2} a_{g,j} \frac{1}{(1-\psi^5)^j}\,. \tag{2.30}$$

Here $[\cdot]$ denotes the Gauss symbol. The coefficients $b_{g,j}$ are determined to cancel the singularity at $\psi = 0$. On the other hand, coefficients $a_{g,j}$ are ambiguities and should be determined by other information.

The expression of a partition function in the t-frame is given by

$$F_g^{\text{A-model}} = \lim_{\bar{t}\to\infty} \left(\frac{(2\pi i)^3}{5^7} \omega_0^2 \right)^{g-1} F_g\,. \tag{2.31}$$

By expanding $F_g^{\text{A-model}}$ in terms of e^{-t}, we will obtain the instanton expansion of the A-model.

3. Generators for the higher genus amplitudes

In this section, we will show that the higher genus amplitudes are expressed as polynomials of finite number of generators. First, in subsection 3.1 we introduce infinite number of generators, and show that the amplitudes can be written as polynomials of these generators. Second, in subsection 3.2, these infinite number of generators turn out to be written as polynomials of finite number of generators. Finally, in section 3.3, we will reconsider the holomorphic anomaly equation and show that the number of generators for partition functions reduces by one. We also state the final form of the claim in the t-frame.

3.1 Expression of amplitudes by infinite number of generators

Let us introduce some notations.

$$
\begin{aligned}
A_p &:= \frac{(\psi\partial_\psi)^p G_{\psi\bar{\psi}}}{G_{\psi\bar{\psi}}}\,, \qquad & B_p &:= \frac{(\psi\partial_\psi)^p e^{-K}}{e^{-K}}\,, \qquad & (p = 1,2,3,\dots)\,,\\
C &:= C_{\psi\psi\psi}\psi^3\,, & X &:= \frac{1}{1-\psi^5}. &
\end{aligned}
\tag{3.1}
$$

Especially, $A := A_1$ and $B := B_1$ are "connections" as

$$
A := A_1 = -\psi\Gamma^{\psi}_{\psi\psi}\,, \qquad B := B_1 = -\psi\partial_\psi K\,. \tag{3.2}
$$

We also denote amplitudes in "(Yukawa coupling)$= 1$ frame" by

$$
P_g := C^{g-1}F_g\,, \qquad P_g^{(n)} = C^{g-1}\psi^n C^{(g)}_{\psi^n}\,, \tag{3.3}
$$

where P_g is defined for $g \geq 2$ and $P_g^{(n)}$ is defined for $g = 0$ and $n \geq 3$, $g = 1$ and $n \geq 1$, or $g \geq 2$ and $n \geq 0$. The first thing we want to show is

Proposition 1. *Each $P_g^{(n)}$ is an degree $(3g-3+n)$ inhomogeneous polynomial of A_p, B_p, X, $(p = 1,2,3,\dots)$, where we assign "degree" p to A_p and B_p, and 1 to X.*

Now, we prove this statement. As preliminaries, we consider two things: the derivatives of generators and the expression of propagators. The derivatives of the quantities of eq. (3.1) becomes

$$
\psi\partial_\psi A_p = A_{p+1} - AA_p\,, \quad \psi\partial_\psi A_p = A_{p+1} - AA_p\,, \quad \psi\partial_\psi X = 5X(X-1)\,, \quad \psi\partial_\psi C = 5XC\,. \tag{3.4}
$$

We find two facts from these equations. First, if $h(A_p, B_p, X)$ is a polynomial of A_p, B_p, X, then the derivative $\psi\partial_\psi h$ is again a polynomial of A_p, B_p, X. Second, the derivative $\psi\partial_\psi$ increases the degree by 1 in general. We can derive the similar facts for the covariant derivative ψD_ψ. Let h be a section of $(T^*)^\ell \otimes L^m$, and assume $\psi^\ell C^{-m/2}h$ is a polynomial of A_p, B_p, X of degree n. Then the covariant derivative (2.19) of h becomes

$$
C^{-m/2}\psi^{\ell+1}D_\psi h = \psi\partial_\psi(\psi^\ell C^{-m/2}h) + \left[-\ell(A+1) - m\left(B - \frac{5}{2}X\right)\right](\psi^\ell C^{-m/2}h)\,, \tag{3.5}
$$

and therefore $C^{-m/2}\psi^{\ell+1}D_\psi h$ is a polynomial of A_p, B_p, X of degree $(n+1)$.

We can write the propagators in eqs. (2.21) in terms of A_p, B_p, X as

$$
T^{\psi\psi} := \frac{C}{\psi^2}S^{\psi\psi} = 2 - 2B - A\,, \qquad T^{\psi} := \frac{C}{\psi}S^{\psi} = 2 - 3B + B_2\,, \tag{3.6}
$$

$$
\begin{aligned}
T := CS = \frac{1}{2}\big[&2 + 2A + A_2 - 3B - A_2B - B^2 - 2AB^2 + 2B^3 + 4B_2 -\\
&- 5BB_2 + B_3 - 5AX - 5BX + 5ABX + 10B^2X - 5B_2X\big]\,.
\end{aligned}
\tag{3.7}
$$

These equations explicitly shows that $T^{\psi\psi}, T^{\psi}, T$ are inhomogeneous polynomials of degree $1, 2, 3$ respectively.

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Let us prove proposition 1 by induction. If we assume $P_g^{(n)}$ is a polynomial of A_p, B_p, X of degree $3g-3+n$, then $P_g^{(n+1)}$ can be written as

$$P_g^{(n+1)} = \psi\partial_\psi P_g^{(n)} + \left[-n(A+1) - (2-2g)(B-\frac{5}{2}X)\right] P_g^{(n)}\,, \tag{3.8}$$

and $P_g^{(n+1)}$ turn out to be a polynomial of degree $(3g-3+n+1)$. As for $g=0$, because $P_{g=0}^{(3)} = 1$ by definition, we can conclude that each $P_{g=0}^{(n)}$, $n = 3,4,5,\dots$ is a polynomial of degree $(-3+n)$. In the case of $g=1$, eq. (2.16) reads

$$P_{g=1}^{(1)} = \frac{31}{3}(1-B) + \frac{5}{12}(X-1) - \frac{1}{2}A\,, \tag{3.9}$$

and we also find that each $P_{g=1}^{(n)}$, $(n = 1,2,3,\dots)$ is a polynomial of degree n.

Let us fix $g \geq 2$ and assume each $P_r^{(n)}$, $r < g$ is a polynomial of degree $(3r-3+n)$. In order to show P_g to be a polynomial, we pick up a diagram $\mathcal{D}$ which contribute to F_g. We denote the number of vertices by k, the number of solid lines by e, the number of half-dashed lines by e', and the number of dashed lines by e'' in the diagram $\mathcal{D}$. We label each vertex by j, $(j = 1,\dots,k)$ and let the genus of the vertex be g_j. We also let n_j solid lines and m_j dashed lines end on the j-th vertex. Then considering the number of lines, we find the relations

$$\sum_{j=1}^{k} n_j = 2e + e'\,, \qquad \sum_{j=1}^{k} m_j = e' + 2e''\,. \tag{3.10}$$

Since $\mathcal{D}$ contribute to F_g, we obtain the relation by counting the order of λ.

$$\sum_{j=1}^{k} (g_j - 1) + e + e' + e'' = g - 1\,. \tag{3.11}$$

By using these relations and the expressions of vertices and propagators (2.26), (3.7), $\mathcal{D}$ is evaluated as

$$\begin{aligned}\mathcal{D} &= (\text{constant}) \times \lambda^{2g-2} \left(\prod_{j=1}^{k} \tilde{C}^{(g_j)}_{\psi^{n_j},\varphi^{m_j}}\right) (S^{\psi\psi})^e (S^\psi)^{e'} (S)^{e''} \\ &= (\text{constant}) \times \lambda^{2g-2} \frac{1}{C^{g-1}} \left(\prod_{j=1}^{k} P_{g_j}^{(n_j)}\right) (T^{\psi\psi})^e (T^\psi)^{e'} (T)^{e''}\,.\end{aligned} \tag{3.12}$$

$T^{\psi\psi}, T^\psi, T$ are polynomials of A_p, B_p, X because of eqs. (3.7), and $P_{r_j}^{(n_j)}$'s are also polynomials due to the assumption of induction. Consequently, $C^{g-1}\mathcal{D}$ is a polynomial. Its degree is evaluated by using eqs. (3.10),(3.11) as [1]

$$\sum_{j=1}^{k} (3g_j - 3 + n_j) + e + 2e' + 3e'' = 3g - 3\,. \tag{3.13}$$

As a result, we can conclude that $C^{g-1}\mathcal{D}$ is a polynomial of A_p, B_p, X of degree $(3g-3)$.

[1] Actually, we need a special care to the vertex with $g_j = 1, n_j = 0$. The easiest way is to set $P_{g=1}^{(n=0)} = 1$ temporally. The statement itself is correct.

So far, we have shown that the Feynman diagram part of P_g is a degree $(3g-3)$ polynomial. Now let us turn to the holomorphic part. Eq. (2.30) is written by using X as

$$f_g(\psi) = \sum_{j=1}^{[(2g-2)/5]} b_{g,j} \left(\frac{X}{X-1}\right)^j + \sum_{j=0}^{2g-2} a_{g,j} X^j \,. \tag{3.14}$$

Actually C can be written as $C = \frac{(2\pi i)^3}{5^3}(X-1)$ because of the explicit form of the $C_{\psi\psi\psi}$ in (2.12). Consequently, we can conclude that $C^{g-1}f_g$ is a degree $(3g-3)$ polynomial of X. Here, we have proved proposition 1.

3.2 Relation between generators

In this subsection, we will show that among the generators in (3.1), A_p $(p = 2, 3, 4, \dots)$ and B_p, $(p = 4, 5, 6, \dots)$ are written as polynomials of A, B, B_2, B_3, X. If we combine this fact and proposition 1, we can conclude that each $P_g^{(n)}$ is a degree $(3g-3+n)$ polynomial of A, B, B_2, B_3, X.

First, let us begin with B_p. By using eq. (2.8) and the definition (3.1), we can write B_p in the following form.

$$B_p = \frac{\Pi^\dagger \Sigma (\psi\partial_\psi)^p \Pi}{\Pi^\dagger \Sigma \Pi} \,. \tag{3.15}$$

Since each component of Π is a period, Π satisfies the Picard-Fuchs equation (2.4)

$$\left\{ (\psi\partial_\psi)^4 - \psi^{-5} (\psi\partial_\psi - 1)(\psi\partial_\psi - 2)(\psi\partial_\psi - 3)(\psi\partial_\psi - 4) \right\} \Pi = 0 \,. \tag{3.16}$$

This equation reads the relation between generators

$$B_4 = 10XB_3 - 35XB_2 + 50XB - 24X \,. \tag{3.17}$$

If we differentiate (3.17) and use the relation $\psi\partial_\psi B_p = B_{p+1} - BB_p$ and (3.17) recursively, we will obtain the expressions of B_p, $p = 4, 5, 6, \dots$ in terms of polynomials of B, B_2, B_3, X of appropriate degrees.

Next, we turn to A_p, $(p = 2, 3, 4, \dots)$. We can rewrite one of the special geometry relation $\partial_\psi \overline{C}_{\bar\psi\bar\psi\bar\psi} = 0$ by using the first equation of (2.22), and the definition of A, B (3.1) as

$$2B\bar\partial_{\bar\psi} S^{\psi\psi} + 2A\bar\partial_{\bar\psi} S^{\psi\psi} + \bar\partial_{\bar\psi}(\psi\partial_\psi S^{\psi\psi}) = 0 \,. \tag{3.18}$$

Moreover, multiply C/ψ^2 this equation and use eq.(3.7), then we obtain the following differential equation

$$-2A\bar\partial_{\bar\psi} B + \bar\partial_{\bar\psi}\left[-2B^2 - A^2 - 2AB + \frac{C}{\psi}\partial_\psi S^{\psi\psi}\right] = 0 \,. \tag{3.19}$$

The last term inside the $\bar\partial_{\bar\psi}$ can be expressed in terms of $A, B, \dots$

$$\frac{C}{\psi}\partial_\psi S^{\psi\psi} = -2B_2 + 2B^2 - A_2 + A^2 - (5X-2)(2-2B-A) \,. \tag{3.20}$$

We can also derive the following relation from the definition (3.1)

$$-A\bar{\partial}_{\bar{\psi}}B = \bar{\partial}(-B_2 + B + B^2)\,. \tag{3.21}$$

If we put these things into eq. (3.19), we obtain the differential equation

$$\bar{\partial}_{\bar{\psi}}(-4B_2 - A_2 - 2AB - 2B + 2B^2 - 2A + 10XB + 5XA) = 0\,. \tag{3.22}$$

We can fix the "holomorphic ambiguity" by asymptotic behavior, and obtain the relation

$$A_2 = -4B_2 - 2AB - 2B + 2B^2 - 2A + 10XB + 5XA - 5X - 1\,. \tag{3.23}$$

If we differentiate (3.23), and use (3.17) and (3.23) recursively, we will obtain the expression of $A_p,\ p = 2, 3, 4, \ldots$ as polynomials of A, B, B_2, B_3, X of appropriate degrees.

3.3 Back to holomorphic anomaly equation

Now, we have shown that each $P_g^{(n)}$ is a degree $(3g-3+n)$ polynomial of A, B, B_2, B_3, X. In this subsection, we will rewrite the holomorphic anomaly equation (2.20), and see the nature of the polynomial $P_g(A, B, B_2, B_3, X)$. As we will see, P_g depends on some special combinations of A, B, B_2, B_3, X.

First, we multiply C^{g-1} both side of eq.(2.20) and see the left-hand side. Since X is holomorphic, the anti-holomorphic derivative of P_g becomes

$$\bar{\partial}_{\bar{\psi}}P_g = \bar{\partial}_{\bar{\psi}}A\frac{\partial P_g}{\partial A} + \bar{\partial}_{\bar{\psi}}B\frac{\partial P_g}{\partial B} + \bar{\partial}_{\bar{\psi}}B_2\frac{\partial P_g}{\partial B_2} + \bar{\partial}_{\bar{\psi}}B_3\frac{\partial P_g}{\partial B_3}\,. \tag{3.24}$$

As we have seen in eq. (3.21), $\bar{\partial}_{\bar{\psi}}B_2$ can be written as

$$\bar{\partial}_{\bar{\psi}}B_2 = (A + 1 + 2B)\bar{\partial}_{\bar{\psi}}B\,. \tag{3.25}$$

Similarly, we can rewrite $\bar{\partial}_{\bar{\psi}}B_3$ as

$$\bar{\partial}_{\bar{\psi}}B_3 = \{(B + 5X)(1 + A + 2B) - B_2 - 10X\}\bar{\partial}_{\bar{\psi}}B\,. \tag{3.26}$$

If we put these things into eq. (2.20) and use the first equation of (2.22) and eq. (3.7), the holomorphic anomaly equation can be written as

$$\begin{aligned}\bar{\partial}_{\bar{\psi}}A\frac{\partial P_g}{\partial A} + \bar{\partial}_{\bar{\psi}}B\left[\frac{\partial P_g}{\partial B} + (A+1+2B)\frac{\partial P_g}{\partial B_2} + \{(B+5X)(1+A+2B) - B_2 - 10X\}\frac{\partial P_g}{\partial B_3}\right] &= \\ = \frac{1}{2}(-\bar{\partial}_{\bar{\psi}}A - 2\bar{\partial}_{\bar{\psi}}B)\left(P_{g-1}^{(2)} + \sum_{r=1}^{g-1}P_r^{(1)}P_{g-r}^{(1)}\right)&\end{aligned} \tag{3.27}$$

If we assume $\bar{\partial}_{\bar{\psi}}A$ and $\bar{\partial}_{\bar{\psi}}B$ are independent, eq. (3.27) yields two independent differential equations. One of these is written as

$$\left[-2\frac{\partial}{\partial A} + \frac{\partial}{\partial B} + (A + 1 + 2B)\frac{\partial}{\partial B_2} + \{(B + 5X)(1 + A + 2B) - B_2 - 10X\}\frac{\partial}{\partial B_3}\right]P_g = 0\,. \tag{3.28}$$

This differential equation gives a constraint for the partition function P_g. To see this, it is convenient to change the variables from (A, B, B_2, B_3, X) to (u, v_1, v_2, v_3, X) as

$$\begin{aligned} u &= B\,, \qquad v_1 = A+1+2B\,, \qquad v_2 = B_2 - B(A+1+2B)\,,\\ v_3 &= B_3 - B\{B(1+A+2B) - B_2 + 5X(1+A+2B) - 10X\}\,, \end{aligned} \tag{3.29}$$

or

$$B = u\,, \qquad A = v_1 - 1 - 2u\,, \qquad B_2 = v_2 + uv_1\,, \tag{3.30}$$

$$B_3 = v_3 + u(-v_2 + 5X(v_1-2))\,. \tag{3.31}$$

In variables (u, v_1, v_2, v_3, X), eq. (3.28) simplifies to

$$\frac{\partial P_g}{\partial u} = 0\,. \tag{3.32}$$

As a result, we can conclude that P_g is independent of u in the valuable (u, v_1, v_2, v_3, X). We summarize this result as the following proposition.

Proposition 2. *Each P_g, $g = 2,3,4,\ldots$ is a degree $(3g-3)$ inhomogeneous polynomial of v_1, v_2, v_3, X, where we assign the degree $1,2,3,1$ for v_1, v_2, v_3, X, respectively.*

Finally, we state proposition 2 in the A-model picture. Recall that the partition function $F_g^{\text{A-model}}$ in A-model picture is related to P_g by

$$F_g^{\text{A-model}} = \lim_{\bar t\to\infty}\left(\frac{(2\pi i)^3\omega_0^2}{5^7 C}\right)^{g-1} P_g\,. \tag{3.33}$$

Therefore, if we define

$$W_1 := \left(\frac{(2\pi i)^3\omega_0^2}{5^7 C}\right)^{1/3}, \qquad V_j := \lim_{\bar t\to\infty} v_j W_1^j,\ (j=1,2,3)\,, \qquad Y_1 := XW_1\,, \tag{3.34}$$

then the final form of the claim is obtained as the theorem.

Theorem 1. *Each $F_g^{\text{A-model}}$, $g = 2,3,\ldots$ is a degree $(3g-3)$ quasi-homogeneous polynomial of V_1, V_2, V_3, W_1, Y_1, where we assign the degree $1,2,3,1,1$ for V_1, V_2, V_3, W_1, Y_1, respectively.*

We write the summary of the final form of generators V_1, V_2, V_3, W_1, Y_1 here. We use the fact that $\bar\psi \to \infty$, $G_{\psi\bar\psi} \propto \bar\psi^{-2}\partial_\psi t$ and $e^{-K} \to \omega_0$ in the limit $\bar t \to \infty$. The function $\omega_0(\psi)$ and $t(\psi)$ is as written in eq. (2.5) and eq. (2.9) respectively. The generators are expressed as

$$\begin{aligned} W_1 &= \left(\frac{\omega_0^2(\psi^{-5}-1)}{5^4}\right)^{1/3}, \qquad Y_1 = W_1\frac{1}{1-\psi^5}\,,\\ V_1 &= W_1\left(\frac{(\psi\partial_\psi)^2 t}{\psi\partial_\psi t} + 2\frac{\psi\partial_\psi\omega_0}{\omega_0}\right), \qquad V_2 = W_1^2\frac{(\psi\partial_\psi)^2\omega_0}{\omega_0} - W_1V_1\frac{\psi\partial_\psi\omega_0}{\omega_0}\,,\\ V_3 &= W_1^3\frac{(\psi\partial_\psi)^3\omega_0}{\omega_0} - W_1\frac{\psi\partial_\psi\omega_0}{\omega_0}(-V_2 + 5Y_1V_1 - 10W_1Y_1)\,. \end{aligned} \tag{3.35}$$

To obtain the instanton expansion, we need to write down the inverse relation $\psi = \psi(t)$ as a power series of e^{-t} and insert it to the above expressions.

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4. Some results for the coefficients of the polynomial representation

So far, we have proved that P_g is a polynomial of v_1, v_2, v_3, X. In this section, we try to determine the coefficients of the polynomial. To do this, the most serious problem is the holomorphic ambiguity. As for some lower genus, say genus 2,3, and 4, we can fix the ambiguity by known results [2, 4]. There are also a part of the coefficients which do not suffer from the ambiguity. We will calculate some of these coefficients for all genus.

In this section, we use proposition 2 form of P_g. In order to get theorem 1 form of $F_g^{\text{A-model}}$, replace v_j with V_j and X with Y_1, and adjust the degree with W_1.

4.1 Lower genus partition functions

We can calculate the coefficients of the polynomial by holomorphic anomaly equation or equivalently the Feynman rule. We should fix the holomorphic ambiguity at each order. For example, the genus 2 partition function can be written in the polynomial form

$$\begin{aligned} P_2 = & \frac{3125}{144} - \frac{15625}{288}v_1 + \frac{125}{24}v_1^2 - \frac{5}{24}v_1^3 - \frac{3125}{36}v_2 + \frac{25}{6}v_1v_2 + \frac{350}{9}v_3 - \frac{28795}{144}X - \\ & - \frac{835}{144}v_1X + \frac{5}{6}v^2X - \frac{2375}{12}v_2X + \frac{205}{144}X^2 - \frac{325}{288}v_1X^2 + \frac{25}{48}X^3 \,. \end{aligned} \tag{4.1}$$

We can also write the genus 3 and 4 partition function in the polynomial form, and show them in appendix A.

4.2 Coefficients of v_3^n

We can calculate some simple part of the coefficients in the full order. In this subsection, we consider the coefficients of v_3^n term. First, we define the following partition function

$$Z(\lambda, v_1, v_2, v_3, X) = \exp\left(\sum_{g=2}^{\infty} \lambda^{2g-2} P_g(v_1, v_2, v_3, X)\right) . \tag{4.2}$$

The holomorphic anomaly equation can be written in the simple form as explained in [2]

$$\begin{aligned} \bar\partial_{\bar\psi} Z \ &= \ \frac{1}{2}\lambda^2(-\bar\partial_{\bar\psi}A - 2\bar\partial_{\bar\psi}B)\left[(P_1^{(2)} + (P_1^{(1)})^2)Z + 2P_1^{(1)}\psi D_\psi Z + \psi^2 D_\psi^2 Z\right] , \\ \psi D_\psi Z \ &:= \ \psi\partial_\psi Z + \left(u - \frac{5}{2}X\right)\lambda\partial_\lambda Z \,. \end{aligned} \tag{4.3}$$

The both side of this equation become quadratic in u. If we use explicit form of $P_1^{(1)}$ and $P_1^{(2)}$, and compare each coefficients of u, we obtain three partial differential equation of Z

$$\begin{aligned} -\frac{2}{\lambda^2}\frac{\partial Z}{\partial v_1} = & \ (\psi\partial_\psi)^2 Z + \frac{25}{4}X^2(\lambda\partial_\lambda)^2 Z - 5X\psi\partial_\psi\lambda\partial_\lambda Z + \\ & + \left(-2v_1 + \frac{5}{6}X - \frac{25}{2}\right)\psi\partial_\psi Z + \\ & + \left(v_2 + 5v_1X - \frac{175}{12}X^2 + \frac{175}{4}X\right)\lambda\partial_\lambda Z + \\ & + \left(\frac{15625}{144} - \frac{125}{6}v_1 + \frac{5}{4}v_1^2 - \frac{25}{3}v_2 + \frac{835}{72}X - \frac{10}{3}v_1X + \frac{325}{144}X^2\right) Z, \end{aligned} \tag{4.4}$$

$$\frac{2}{\lambda^2}\left(\frac{\partial Z}{\partial v_2}+5X\frac{\partial Z}{\partial v_3}\right) = -\frac{50}{3}\psi\partial_\psi Z+\left(\frac{85}{2}X-v_1-\frac{25}{2}\right)\lambda\partial_\lambda Z+2\psi\partial_\psi\lambda\partial_\lambda Z- \\ -5X(\lambda\partial_\lambda)^2 Z+\left(-\frac{3125}{18}+\frac{25}{3}v_1-\frac{125}{18}X\right)Z\,, \tag{4.5}$$

$$\frac{2}{\lambda^2}\frac{\partial Z}{\partial v_3} = \left(\frac{\chi}{12}-1\right)\lambda\partial_\lambda Z+(\lambda\partial_\lambda)^2 Z+\frac{\chi}{24}\left(\frac{\chi}{24}-1\right)Z\,. \tag{4.6}$$

Here $\psi\partial_\psi$ act to Z as

$$\psi\partial_\psi Z = (-v_1^2-2v_2-10X+5v_1X)\frac{\partial Z}{\partial v_1}+(-v_1v_2+v_3)\frac{\partial Z}{\partial v_2}+ \\ +(v_2^2-24X-25v_2X-5v_1v_2X+10v_3X)\frac{\partial Z}{\partial v_3}+5X(X-1)\frac{\partial Z}{\partial X}\,. \tag{4.7}$$

Now, in order to see only the v_3 and λ dependence, we define a function

$$\widetilde{Z}(\lambda,v_3) := Z(\lambda,v_1=0,v_2=0,v_3,X=0)\,. \tag{4.8}$$

The coefficients of v_3^n terms are encoded in this function $\widetilde{Z}$. This function $\widetilde{Z}$ satisfies also the differential equation (4.6). As a result, $\widetilde{Z}$ is solved by the formal power series as

$$\widetilde{Z} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\lambda^{2n}v_3^k\frac{\Gamma\left(\frac{\chi}{24}-1+2n\right)}{2^k k!\Gamma\left(\frac{\chi}{24}-1+2n-2k\right)}\alpha_{n-k}\,, \tag{4.9}$$

where the constants α_ℓ, $(\ell=0,1,2,\dots)$ are part of the holomorphic ambiguities. These are fixed by considering the constant map contribution [9, 10, 11], namely

$$\lim_{t\to\infty}F_g^{\text{A-model}} = \frac{(-1)^g B_g B_{g-1}}{4g(2g-2)(2g-2)!}\chi\,, \tag{4.10}$$

where $B_g,\ g=1,2,3,\dots$ are the Bernoulli numbers. We also use the fact that in the limit $\bar{t}\to\infty$ and $t\to\infty$, v_j and X vanish. The α_n are expressed as

$$\widetilde{Z}(\lambda,v_3=0) = \sum_{n=0}^{\infty}\lambda^{2n}\alpha_n = \exp\left(\sum_{g=2}^{\infty}\lambda^{2g-2}\frac{(-1)^g B_g B_{g-1}}{4g(2g-2)(2g-2)!}(-5^4\chi)\right)\,. \tag{4.11}$$

Let us make a remark here. We denote the generating function of the coefficient of $\lambda^{2n}v_3^n$ in (4.9) by $\widetilde{Z}^{(0)}$. The explicit form of $\widetilde{Z}^{(0)}$ can be written as a formal series

$$\widetilde{Z}^{(0)}(\lambda^2 v_3) = \sum_{n=0}^{\infty}(\lambda^2 v_3)^n\frac{\Gamma\left(\frac{\chi}{24}-1+2n\right)}{2^n n!\Gamma\left(\frac{\chi}{24}-1\right)}. \tag{4.12}$$

This series can be rewritten as the asymptotic expansion of Kummer confluent hypergeometric function ${}_1F_1(\alpha,\gamma;z)$. We can write

$$\widetilde{Z}^{(0)}(\lambda^2 v_3) = C_1(2\lambda^2 v_3)^{-\frac{1}{2}\left(\frac{\chi}{24}-1\right)}{}_1F_1\left(\frac{1}{2}\left(\frac{\chi}{24}-1\right),\frac{1}{2};-\frac{1}{2\lambda^2 v_3}\right)+ \\ +C_2(2\lambda^2 v_3)^{-\frac{\chi}{48}}{}_1F_1\left(\frac{\chi}{48},\frac{3}{2};-\frac{1}{2\lambda^2 v_3}\right)\,, \tag{4.13}$$

where C_1 and C_2 are constants which satisfies

$$\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1-\frac{\chi}{48}\right)}C_1+\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-\frac{\chi}{48}\right)}C_2=1\,. \tag{4.14}$$

The expression (4.13) might give some non-perturbative information of the topological string theory.

5. Conclusion and discussion

In this paper, we have shown that the topological partition functions of the quintic can be written as polynomials of five generators. We have written down the polynomial forms of F_2, F_3, F_4. We also obtain the coefficients of v_3^n for all genus.

To fix the holomorphic ambiguity is the most serious problem to obtain the coefficients of the polynomial. One possible way to do this is using the heterotic dual description [9] and [12]–[15]. Also the large N duality [16] might give some hints.

The fact that F_g's are polynomials of five generators implies that there are polynomial relations between F_g's. In other words, for $2 \le g_1 < g_2 < \cdots < g_k$, $k \ge 6$, there is a quasi-homogeneous polynomial $Q(F_{g_1}, \ldots, F_{g_k})$ such that

$$Q(F_{g_1}, \ldots, F_{g_k}) = 0\,. \tag{5.1}$$

These polynomial relations are completely gauge invariant. Therefore we can expect some physical or mathematical meaning of the coefficients of this polynomial. If this meaning becomes clear, it might be useful to fix the holomorphic ambiguity.

In this paper, we mainly treat the quintic hypersurface. We can also do the similar analysis for the Calabi-Yau hypersurfaces in weighted projective spaces treated in [6]. See appendix B. The generalization to other Calabi-Yau manifolds, especially complete intersection in products of weighted projective spaces [17, 18] is a future problem.

Acknowledgments

We would like to thank Jun Li, Bong H. Lian, Kefeng Liu, Hirosi Ooguri, and Cumrun Vafa for useful discussions. This work was supported in part by NSF grants DMS-0074329 and DMS-0306600.

A. Polynomial form of genus 3 and 4 partition function

Here we show the genus 3 and 4 partition functions in the polynomial form. We use the result of [4] to fix the ambiguity.

$$\begin{aligned} P_3 = \frac{5}{72576}\big(&781250 - 2734375 v_1^3 + 787500 v_1^4 - 94500 v_1^5 + 4536 v_1^6 + \\ &+ 6562500 v_2 - 16721250 v_1 v_2 - 2625000 v_1^2 v_2 + 819000 v_1^3 v_2 - \\ &- 54432 v_1^4 v_2 - 18112500 v_2^2 - 1772400 v_1 v_2^2 + 295344 v_1^2 v_2^2 - \end{aligned}$$

$$
\begin{aligned}
&- 936320 v_2^3 - 4935000 v_3 + 12337500 v_1 v_3 - 1184400 v_1^2 v_3 + \\
&+ 47376 v_1^3 v_3 + 19740000 v_2 v_3 - 947520 v_1 v_2 v_3 - 4421760 v_3^2 + \\
&+ 27683000 X - 72635850 v_1 X + 12252135 v_1^2 X - 3366615 v_1^3 X + \\
&+ 604044 v_1^4 X - 41580 v_1^5 X - 81544680 v_2 X - 54284034 v_1 v_2 X + \\
&+ 3202584 v_1^2 v_2 X + 93240 v_1^3 v_2 X - 99165864 v_2^2 X + 3824016 v_1 v_2^2 X + \\
&+ 45473064 v_3 X + 1318632 v_1 v_3 X - 189504 v_1^2 v_3 X + 45007200 v_2 v_3 X - \\
&- 112828006 X^2 - 12527550 v_1 X^2 + 5722185 v_1^2 X^2 - 1658685 v_1^3 X^2 + \\
&+ 176400 v_1^4 X^2 - 233375520 v_2 X^2 - 3865134 v_1 v_2 X^2 + 104160 v_1^2 v_2 X^2 - \\
&- 113818740 v_2^2 X^2 - 323736 v_3 X^2 + 256620 v_1 v_3 X^2 + 3339968 X^3 - \\
&- 4795350 v_1 X^3 + 2353785 v_1^2 X^3 - 444325 v_1^3 X^3 + 819840 v_2 X^3 - \\
&- 266910 v_1 v_2 X^3 - 118440 v_3 X^3 + 1696500 X^4 - 1683150 v_1 X^4 + \\
&+ 686175 v_1^2 X^4 + 119700 v_2 X^4 + 477000 X^5 - \\
&- 598500 v_1 X^5 + 225000 X^6 \Big) \,. \qquad \text{(A.1)}
\end{aligned}
$$

$$
\begin{aligned}
P_4 \;=\; & \frac{1}{850500000000000} \times \\
& \times (476837158203125000 - 1251697540283203125 v_1^5 + 640869140625000000 v_1^6 - 144195556640625000 v_1^7 + \\
& + 16611328125000000 v_1^8 - 815800781250000 v_1^9 + 1907348632812500000 v_2 - 91552734375000000 v_1 v_2 + \\
& + 2002716064453125000 v_1^2 v_2 - 1206703186035156250 0 v_1^3 v_2 + 1625244140625000000 v_1^4 v_2 + \\
& + 307617187500000000 v_1^5 v_2 - 88593750000000000 v_1^6 v_2 + 6157265625000000 v_1^7 v_2 + \\
& + +1001358032226562500 0 v_2^2 - 2375221252441406250 0 v_1 v_2^2 - 1525268554687500000 0 v_1^2 v_2^2 + \\
& + 2514770507812500000 v_1^3 v_2^2 + 55371093750000000 v_1^4 v_2^2 - 20981953125000000 v_1^5 v_2^2 - \\
& - 22117333984375000000 v_2^3 - 9664306640625000000 v_1 v_2^3 + 970429687500000000 v_1^2 v_2^3 + \\
& + 272409375000000000 v_1^3 v_2^3 - 3177453125000000000 v_2^4 + 384037500000000000 v_1 v_2^4 - \\
& - 4501342773437500000 v_3 - 4005432128906250000 v_1 v_3 + 5391311645507812500 v_1^2 v_3 + \\
& + 3402587890625000000 v_1^3 v_3 - 1176635742187500000 v_1^4 v_3 + 147656250000000000 v_1^5 v_3 - \\
& - 7190859375000000 v_1^6 v_3 - 16918945312500000000 v_2 v_3 + 4310947265625000000 v_1 v_2 v_3 + \\
& + 1896972656250000000 v_1^2 v_2 v_3 - 1176328125000000000 v_1^3 v_2 v_3 + 84223125000000000 v_1^4 v_2 v_3 + \\
& + 41825683593750000000 v_2^2 v_3 + 1608140625000000000 v_1 v_2^2 v_3 - 443480625000000000 v_1^2 v_2^2 v_3 + \\
& + 1499575000000000000 v_2^3 v_3 + 5387402343750000000 v_3^2 - 1346850585937500000 v_1 v_3^2 + \\
& + 1292976562500000000 v_1^2 v_3^2 - 51719062500000000 v_1^3 v_3^2 - 21549609375000000000 v_2 v_3^2 + \\
& + 1034381250000000000 v_1 v_2 v_3^2 + 32180750000000000000 v_3^3 + 3776550292968750000 X + \\
& + 17503967285156250000 v_1 X - 22250582885742187500 v_1^2 X - 2807802368164062500 0 v_1^3 X + \\
& + 14768562445068359375 v_1^4 X - 5377867954101562500 v_1^5 X + 1345452978515625000 v_1^6 X - \\
& - 190480253906250000 v_1^7 X + 11572558593750000 v_1^8 X + 95444091796875000000 v_2 X - \\
& - 22654800659179687500 0 v_1 v_2 X - 2529587658691406250 0 v_1^2 v_2 X - 1433358472656250000 0 v_1^3 v_2 X + \\
& + 3472856542968750000 v_1^4 v_2 X + 372585937500000000 v_1^5 v_2 X - 323367187500000000 v_1^6 v_2 X - \\
& - 143172919189453125000 v_2^2 X - 21667619326171875000 0 v_1 v_2^2 X - 13990677832031250000 v_1^2 v_2^2 X + \\
& + 5559784453125000000 v_1^3 v_2^2 X - 248480859375000000 v_1^4 v_2^2 X - 21744233382812500000 0 v_2^3 X - \\
& - 10364004218750000000 v_1 v_2^3 X + 2042381250000000000 v_1^2 v_2^3 X - 7434783125000000000 v_2^4 X - \\
& - 59099116210937500000 v_3 X + 15281040673828125000 0 v_1 v_3 X - 22033919824218750000 v_1^2 v_3 X + \\
& + 5277016992187500000 v_1^3 v_3 X - 948538828125000000 v_1^4 v_3 X + 66002343750000000 v_1^5 v_3 X + \\
& + 175039331250000000000 v_2 v_3 X + 12234670875000000000 v_1 v_2 v_3 X - 8964263437500000000 v_1^2 v_2 v_3 X + \\
& + 275625000000000000 v_1^3 v_2 v_3 X + 21726759468750000000 v_2^2 v_3 X - 8929869375000000000 v_1 v_2^2 v_3 X - \\
& - 49641680156250000000 v_3^2 X - 1439513906250000000 v_1 v_3^2 X + 2068762500000000000 v_1^2 v_3^2 X - \\
& - 49133109375000000000 v_2 v_3^2 X + 16357976000976562500 0 X^2 - 44331300080566406250 0 v_1 X^2 + \\
& + 10182216008789062500 0 v_1^2 X^2 - 48842653500976562500 v_1^3 X^2 + 19449174848632812500 v_1^4 X^2 -
\end{aligned}
$$

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$$
\begin{aligned}
&- 5720941560058593750v_1^5 X^2 + 1020827636718750000v_1^6 X^2 - 78894580078125000v_1^7 X^2 - \\
&- 30455103978515625000 0v_2 X^2 - 7347367608398437500 00v_1 v_2 X^2 + 8975181468750000000 0v_1^2 v_2 X^2 - \\
&- 18545016943359375000v_1^3 v_2 X^2 + 19853244140625000000v_1^4 v_2 X^2 + 2569218750000000000v_1^5 v_2 X^2 - \\
&- 10241052773437500000 00v_2^2 X^2 - 27689178706054687500 0v_1 v_2^2 X^2 + 13071216210937500000 0v_1^2 v_2^2 X^2 + \\
&+ 6429814453125000000v_1^3 v_2^2 X^2 - 545788993203125000000v_2^3 X^2 + 19217201171875000000v_1 v_2^3 X^2 + \\
&+ 248822582949218750000v_3 X^2 + 236622750000000000000v_1 v_3 X^2 - 94406371582031250000v_1^2 v_3 X^2 + \\
&+ 26190445312500000000v_1^3 v_3 X^2 - 2805345703125000000v_1^4 v_3 X^2 + 507984192187500000000v_2 v_3 X^2 + \\
&+ 104479626562500000000v_1 v_2 v_3 X^2 - 74390859375000000000v_1^2 v_2 v_3 X^2 + 248917708593750000000v_2^2 v_3 X^2 + \\
&+ 35341359375000000000v_3^2 X^2 - 2801449218750000000v_1 v_3^2 X^2 - 396523506113281250000X^3 - \\
&+ 103635926660156250000v_1 X^3 + 73077346025390625000v_1^2 X^3 - 39332128759765625000v_1^3 X^3 + \\
&+ 1442326373291015625 0v_1^4 X^3 - 3344353051757812500v_1^5 X^3 + 342342041015625000v_1^6 X^3 - \\
&- 128131566673828125000 0v_2 X^3 - 94647117539062500000v_1 v_2 X^3 + 33246505810546875000v_1^2 v_2 X^3 - \\
&- 73351755859375000000v_1^3 v_2 X^3 + 2628999023437500000v_1^4 v_2 X^3 - 129391759423828125000 0v_2^2 X^3 - \\
&- 16293676464843750000v_1 v_2^2 X^3 - 12941455078125000000v_1^2 v_2^2 X^3 - 418847288671875000000v_2^3 X^3 - \\
&- 62791721875000000000v_3 X^3 + 83235181640625000000v_1 v_3 X^3 - 37342081054687500000v_1^2 v_3 X^3 + \\
&+ 7080458984375000000v_1^3 v_3 X^3 - 242536875000000000000v_2 v_3 X^3 + 11946703125000000000v_1 v_2 v_3 X^3 + \\
&+ 12929765625000000000v_3^2 X^3 + 290532594824218750000X^4 - 55182304052734375000v_1 X^4 + \\
&+ 464394632812500000000v_1^2 X^4 - 23409994384765625000v_1^3 X^4 + 7303842895507812500v_1^4 X^4 - \\
&- 1051710479736328125v_1^5 X^4 + 244807308593750000000v_2 X^4 - 277932187500000000000v_1 v_2 X^4 + \\
&+ 11772791748046875000v_1^2 v_2 X^4 - 10985718994140625000v_1^3 v_2 X^4 + 29131706542968750000v_2^2 X^4 - \\
&- 4635201416015625000v_1 v_2^2 X^4 - 30679168945312500000v_3 X^4 + 26799301757812500000v_1 v_3 X^4 - \\
&- 1095309448242187500v_1^2 v_3 X^4 - 54239062500000000000v_2 v_3 X^4 + 178891056405618105920X^5 - \\
&- 29737974121093750000v_1 X^5 + 24661093139648437500v_1^2 X^5 - 10760107421875000000v_1^3 X^5 + \\
&+ 238751205444335937 5v_1^4 X^5 + 94093359375000000000v_2 X^5 - 91147375488281250000v_1 v_2 X^5 + \\
&+ 2014674682617187500v_1^2 v_2 X^5 + 1821862792968750000v_2^2 X^5 - 7614228515625000000v_3 X^5 + \\
&+ 9565356445312500000v_1 v_3 X^5 + 78959109356732538160X^6 - 15704566040039062500v_1 X^6 + \\
&+ 10229498291015625000v_1^2 X^6 - 3995840454101562500v_1^3 X^6 + 28025463867187500000v_2 X^6 - \\
&- 18482299804687500000v_1 v_2 X^6 - 3598901367187500000v_3 X^6 + 46484252694680605920X^7 - \\
&- 55862182617187500000v_1 X^7 + 46864929199218750000v_1^2 X^7 + 6834594726562500000v_2 X^7 + \\
&+ 12974853515625000000X^8 - 3417297363281250000v_1 X^8 + 115356445312500000000yX^9\,)\,.
\end{aligned}
\tag{A.2}
$$

B. Generalization to the hypersurfaces in weighted projective space

Here, we write the generators of the amplitudes for the hypersurfaces in weighted projective spaces $k = 6, 8, 10$ in the notation of [6]. The generators $A_p, B_p,\ (p = 1, 2, 3, \dots)$ are defined as the same way as eqs. (3.1). We also define $C = C_{\psi\psi\psi}\psi^3$ as in eq. (3.1). On the other hand, X is defined as

$$
X = \frac{1}{1-\psi^k}\,. \tag{B.1}
$$

The derivatives of these things are written as

$$
\psi\partial_\psi A_p = A_{p+1} - AA_p\,, \quad \psi\partial_\psi A_p = A_{p+1} - AA_p\,, \quad \psi\partial_\psi X = kX(X-1)\,, \quad \psi\partial_\psi C = kXC\,. \tag{B.2}
$$

The relation between generators are modified as follows. Eq.(3.17) is modified as

$$
\begin{aligned}
k &= 6\,, & B_4 &= 12XB_3 - 49XB_2 + 78XB - 40X\,,\\
k &= 8\,, & B_4 &= 16XB_3 - 86XB_2 + 176XB - 105X\,,\\
k &= 10\,, & B_4 &= 20XB_3 - 130XB_2 + 300XB - 189X\,,
\end{aligned}
\tag{B.3}
$$

Eq. (3.23) is modified as

$$A_2 = -4B_2 - 2AB - 2B + 2B^2 - 2A + 2kXB + kXA - 1 - r_k X\,, \tag{B.4}$$

$$r_6 = 7\,, \qquad r_8 = 14\,, \qquad r_{10} = 20\,. \tag{B.5}$$

The u, v_1, v_2, v_3 variables in eq. (3.31) are introduced as

$$\begin{aligned} B &= u\,, \qquad A = v_1 - 1 - 2u\,, \qquad B_2 = v_2 + uv_1\,, \\ B_3 &= v_3 - uv_2 + kXuv_1 - (r_k + k)uX\,. \end{aligned} \tag{B.6}$$

The partition function $P_g := C^{g-1}F_g$ can be written as a degree $(3g-3)$ inhomogeneous polynomial of v_1, v_2, v_3, X. For example, P_2 of $k=6$ hypersurface becomes

$$\begin{aligned} P_2 = &\frac{459}{20} - \frac{441}{8}v_1 + \frac{21}{4}v_1^2 - \frac{5}{24}v_1^3 - \frac{357}{4}v_2 + \frac{17}{4}v_1v_2 + \frac{323}{8}v_3 - \frac{13873}{48}X - 7v_1X + \\ &+ v_1^2X - \frac{493}{2}v_2X + \frac{491}{240}X^2 - \frac{13}{8}v_1X^2 + \frac{9}{10}X^3. \end{aligned} \tag{B.7}$$

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